DEFINITIONS AND THEOREMS ON GROUP ACTIONS
MATH 100A

Remark 1. The following are the definitions given in class about group actions, as well as the statements of the theorems we have proved in class.

Definition 2. Let G be a group and S a set. A *group action* of G on S is a function:

$$G \times S \to S$$
$$(g, s) \mapsto g \cdot s$$

satisfying the following properties:

(1) $e \cdot s = s$ for all $s \in S$, and
(2) $g \cdot (h \cdot s) = (g * h) \cdot s$.

Remark 3. Throughout these notes G will denote a group, and S will denote a set with an action of G on S.

Proposition 4. *A group action of G on S is equivalent to a homomorphism:*

$$\phi \colon G \to A(S).$$

Definition 5. Let $x \in S$. The *orbit of x* is the subset:

$$\mathrm{Orb}(x) = \{g \cdot x | g \in G\} \subset S.$$

Definition 6. The *stabilizer of x* is the subset:

$$\mathrm{Stab}(x) = \{g \in G | g \cdot x = x\} \subset G.$$

Definition 7. The action of G on S is called *transitive* if there exists $x \in S$ such that

$$\mathrm{Orb}(x) = S.$$

Example 8 (Coset Action)**.** Let $H \leq G$ be a subgroup. Let

$$S = \{aH | a \in G\}$$

be the set of left cosets of H in G. There is an action:

$$G \times S \to S$$
$$(g, (aH)) \mapsto g \cdot (aH) := (g * a)H.$$

Example 9 (The Conjugation Action)**.** Let G be a group and let $S = G$. There is an action

$$G \times S \to S$$
$$(g, x) \mapsto g \cdot x := g * x * g^{-1}.$$

If $x \in S$, then $\mathrm{Stab}(x) = C(x) = \{g \in G | gx = xg\}$.

Proposition 10. *The relation on S*

$$x \sim y \iff x = g \cdot y$$

is an equivalence relation. The equivalence classes are the orbits.

Proposition 11. *Let $x \in S$. The stabilizer of x, is a subgroup of G:*

$$\mathrm{Stab}(x) \leq G.$$

Theorem 12 (Orbit-Stabilizer Theorem). *Let G act on S. Let $x \in S$. There is a bijection:*

$$\{\textit{Left cosets of}\ \mathrm{Stab}(x)\} \leftrightarrow \mathrm{Orb}(x).$$

In particular, $|\mathrm{Orb}(x)| = [G : \mathrm{Stab}(x)]$ *and if G is finite then* $|\mathrm{Orb}(x)| = |G|/|\mathrm{Stab}(x)|$.

Corollary 13. *The set S is a disjoint union of its orbits, and if S and G are finite then:*

$$|S| = \sum_{\substack{\textit{distinct orbits} \\ \left(\substack{\textit{choosing one } x \\ \textit{in each orbit}}\right)}} |\mathrm{Orb}(x)| = \sum_{\substack{\textit{distinct orbits} \\ \left(\substack{\textit{choosing one } x \\ \textit{in each orbit}}\right)}} \frac{|G|}{|\mathrm{Stab}(x)|}.$$

Corollary 14 (The Class Equation). *Let G be a finite group, then:*

$$|G| = \sum_{\substack{\textit{conjugacy classes} \\ \left(\substack{\textit{choosing one } x \\ \textit{in each conjugacy class}}\right)}} |\mathrm{cl}(x)| = \sum_{\substack{\textit{conjugacy classes} \\ \left(\substack{\textit{choosing one } x \\ \textit{in each conjugacy class}}\right)}} \frac{|G|}{|C(x)|}.$$

Corollary 15. *Let G be a group with $|G| = p^k$ (where p is a prime and $k \geq 1$). Then $|Z(G)| \geq p$.*

Definition 16. A group G is *simple* if there are no nontrivial normal subgroups in G.

Theorem 17. *A_5 is a simple group.*

Definition 18. Let $g \in G$. The *fixed set* of g is the subset

$$S^g := \{x \in S | g \cdot x = x\} \subset S.$$

Definition 19. The *set of orbits* of the action of G on S is denoted $(G \backslash S)$.

Theorem 20 (Burnside's Lemma). *Let G be a finite group, and S a finite set. The number of orbits is equal to the average size of the fixed sets, i.e.*

$$|(G \backslash S)| = \frac{1}{|G|} \sum_{g \in G} |S^g|.$$

Theorem 21 (Cauchy's Theorem). *Let G be a finite group of order n. If p is a prime number such that $p|n$, then there exists $x \in G$ with $|x| = p$.*